Matching-Criterion for Identifiability in Sparse Factor Analysis

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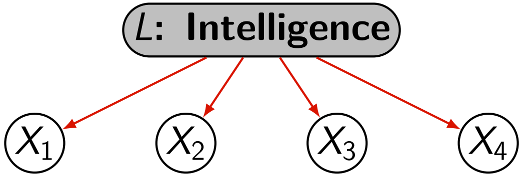
Spearman's One-Factor Model (1904)



Linear structural equations:

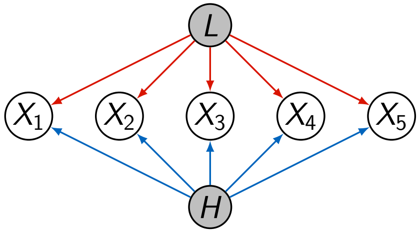
$$
\begin{aligned}
X_1 &= \lambda_{10} + \lambda_{1L} L + \varepsilon_1,\\
X_2 &= \lambda_{20} + \lambda_{2L} L + \varepsilon_2,\\
X_3 &= \lambda_{30} + \lambda_{3L} L + \varepsilon_3,\\
X_4 &= \lambda_{40} + \lambda_{4L} L + \varepsilon_4.
\end{aligned}
$$

Jointly independent errors: $\varepsilon_1, \ldots, \varepsilon_4$.

$\text{Var}[\varepsilon_v] = \omega_{vv} < \infty$, $\text{Var}[L] = 1$.

Topic: Can we recover the "factor loadings" λ_{vL} and the "error variances" ω_{vv} from $\Sigma = \text{Var}[X]$?

Example: Two-Factor Model



$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{pmatrix} = \begin{pmatrix} \lambda_{1L} & \lambda_{1H} \\ \lambda_{2L} & \lambda_{2H} \\ \lambda_{3L} & \lambda_{3H} \\ \lambda_{4L} & \lambda_{4H} \\ \lambda_{5L} & \lambda_{5H} \end{pmatrix} \cdot \begin{pmatrix} L \\ H \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \end{pmatrix}$$

Covariances:

$$\text{Var}[X_v] = \omega_{vv} + \lambda_{vL}^2 + \lambda_{vH}^2 \quad \text{and} \quad \text{Cov}[X_v, X_w] = \lambda_{vL}\lambda_{wL} + \lambda_{vH}\lambda_{wH}.$$

Observed covariance matrix:

$$\Sigma = \begin{pmatrix} \lambda_{1L} & \lambda_{1H} \\ \lambda_{2L} & \lambda_{2H} \\ \lambda_{3L} & \lambda_{3H} \\ \lambda_{4L} & \lambda_{4H} \\ \lambda_{5L} & \lambda_{5H} \end{pmatrix} \begin{pmatrix} \lambda_{1L} & \lambda_{1H} \\ \lambda_{2L} & \lambda_{2H} \\ \lambda_{3L} & \lambda_{3H} \\ \lambda_{4L} & \lambda_{4H} \\ \lambda_{5L} & \lambda_{5H} \end{pmatrix}^\top + \begin{pmatrix} \omega_{11} & 0 & 0 & 0 & 0 \\ 0 & \omega_{22} & 0 & 0 & 0 \\ 0 & 0 & \omega_{33} & 0 & 0 \\ 0 & 0 & 0 & \omega_{44} & 0 \\ 0 & 0 & 0 & 0 & \omega_{55} \end{pmatrix} = \Lambda\Lambda^\top + \Omega.$$

Full Factor Analysis Models

Identifiability problem:

$$\text{For } \Lambda \in \mathbb{R}^{p \times m} \text{ and } \Omega \in \text{diag}^p_+, \text{ does } \Sigma = \Lambda\Lambda^\top + \Omega \text{ have a unique solution?}$$

Observations:

- Rotational indeterminacy if no restrictions on Λ.

$$\Lambda\Lambda^\top + \Omega = (\Lambda Q)(Q^\top \Lambda^\top) + \Omega \quad \text{for all } Q \text{ orthogonal.}$$

- Literature: Identification problem centers around recovering $\Lambda\Lambda^\top$ and Ω. Interpretation of Λ is difficult.

This talk: Λ sparse.

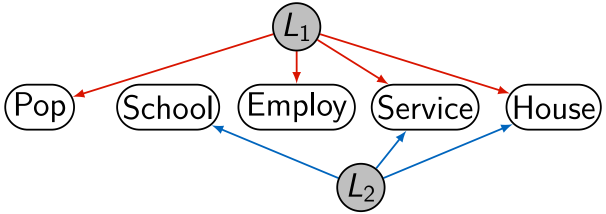
- Often claimed that sparsity “improves” identifiability and interpretability of the factor loadings.

  e.g. Trendafilov et al. (2017), Sparse exploratory factor analysis, or
  Frühwirth-Schnatter et al. (2023) Sparse Bayesian factor analysis when the number of factors is unknown.

- Real-world phenomena: Many observed variables should not depend on all factors.

  Thurstone (1931), Multipe Factor Analysis.

Socio-economic Example from Harmann (1976), Modern Factor Analysis



$$X = \Lambda L + \varepsilon, \quad \text{where } \Lambda = \begin{pmatrix} \lambda_{11} & 0 \\ 0 & \lambda_{22} \\ \lambda_{31} & 0 \\ \lambda_{41} & \lambda_{42} \\ \lambda_{51} & \lambda_{52} \end{pmatrix}.$$

Observed covariance matrix (when $\text{Var}[L] = I$):

$$\Sigma = \Lambda\Lambda^\top + \Omega = \begin{pmatrix} \omega_{11} + \lambda_{11}^2 & 0 & \boxed{\lambda_{11}\lambda_{31}} & \boxed{\lambda_{11}\lambda_{41}} & \lambda_{11}\lambda_{51} \\ 0 & \omega_{22} + \lambda_{22}^2 & 0 & \lambda_{22}\lambda_{42} & \lambda_{22}\lambda_{52} \\ \lambda_{11}\lambda_{31} & 0 & \omega_{33} + \lambda_{31}^2 & \boxed{\lambda_{31}\lambda_{41}} & \lambda_{31}\lambda_{51} \\ \lambda_{11}\lambda_{41} & \lambda_{22}\lambda_{42} & \lambda_{31}\lambda_{41} & \omega_{44} + \lambda_{41}^2 + \lambda_{52}^2 & \lambda_{41}\lambda_{51} + \lambda_{42}\lambda_{52} \\ \lambda_{11}\lambda_{51} & \lambda_{22}\lambda_{52} & \lambda_{31}\lambda_{51} & \lambda_{41}\lambda_{51} + \lambda_{42}\lambda_{52} & \omega_{55} + \lambda_{51}^2 + \lambda_{52}^2 \end{pmatrix}.$$

We see that

1) $\sqrt{\frac{\sigma_{13}\sigma_{14}}{\sigma_{34}}} = \sqrt{\frac{\lambda_{11}\lambda_{31}\,\lambda_{11}\lambda_{41}}{\lambda_{31}\lambda_{41}}} = \sqrt{\lambda_{11}^2} = a_1\lambda_{11}$ with $a_1 \in \{\pm 1\}$ and $\sigma_{34} = \lambda_{31}\lambda_{41} \neq 0$ 'almost surely',

2) $\frac{\sigma_{13}}{\sqrt{\sigma_{13}\sigma_{14}/\sigma_{34}}} = \frac{\lambda_{11}\lambda_{31}}{a_1\lambda_{11}} = a_1\lambda_{31}$ with $\lambda_{11} \neq 0$ 'almost surely'.

$\Longrightarrow$ Can identify $\Lambda_{\text{ch}(L_1),L_1}$ up to column-sign, similarly $\Lambda_{\text{ch}(L_2),L_2}$.

General Setup

Variables:

Observed: $X = (X_v)_{v \in V}$ Latent: $L = (L_h)_{h \in \mathcal{H}}$

Graph:

Bipartite directed graph $G = (V \dot{\cup} \mathcal{H}, D)$, that is, $D \subseteq \mathcal{H} \times V$.

Sparse factor analysis model:

$$X = \Lambda L + \varepsilon$$

– all latent factors and error terms in (L, ε) are mutually independent, so $\Omega = \mathrm{Var}[\varepsilon] = \mathrm{diag}(\omega_{vv} : v \in V)$ diagonal, and $\mathrm{Var}[L] = I$.

– parameter matrix Λ is sparse and supported over edge set D (write $\Lambda \in \mathbb{R}^D$).

Content of the Talk

Definition

Every factor analysis graph G yields a parametrization of the observed covariance matrix:

$$\tau_G : (\Lambda, \Omega) \longmapsto \Sigma \equiv \Lambda\Lambda^\top + \Omega.$$

Fiber: $\mathcal{F}_G(\Omega, \Lambda) = \{(\widetilde{\Omega}, \tilde{\Lambda}) : \tau_G(\widetilde{\Omega}, \tilde{\Lambda}) = \tau_G(\Omega, \Lambda)\}$.

The model given by G is generically sign-identifiable if

$$\mathcal{F}_G(\Omega, \Lambda) = \{(\widetilde{\Omega}, \tilde{\Lambda}) : \widetilde{\Omega} = \Omega \text{ and } \tilde{\Lambda} = \Lambda\Psi \text{ for } \Psi \in \{\pm 1\}^{|\mathcal{H}| \times |\mathcal{H}|} \text{ diagonal}\} \qquad \text{for 'almost all' } (\Lambda, \Omega).$$

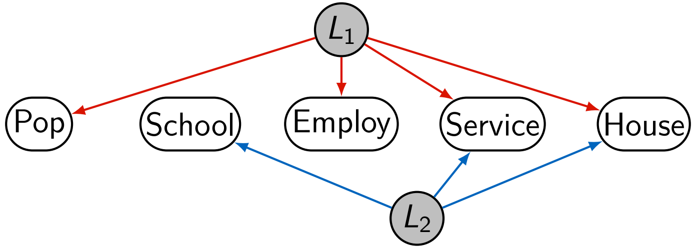
Main Contribution:

- Sufficient graphical condition for generic sign-identifiability.
- Recursive polynomial time algorithm.
  (caveat: polynomial time when bounding a cardinality in a search step)

Gröbner basis computations solve the problem ... on a small scale.

Our Software

```
# Define graph
> L = matrix(c(1, 0,
+              0, 1,
+              1, 0,
+              1, 1,
+              1, 1), 5, 2, byrow=TRUE)
> g = FactorGraph(L)
>
> # Check identifiability
> res = mID(g)
Generic Sign-Identifiability Summary
# nr. of latent nodes that are gen. sign-identifiable: 2
# gen. sign-identifiable nodes: 1, 2
```



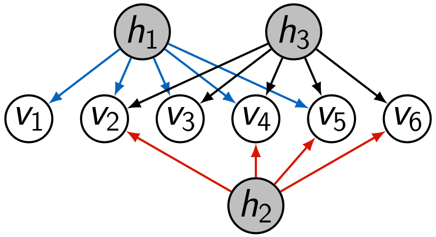
R-package SEMID available on CRAN.

Zero Upper Triangular Assumption

Definition

The graph G satisfies the Zero Upper Triangular Assumption (ZUTA) if there exists an ordering $\prec$ on the latent nodes $\mathcal{H}$ such that $\text{ch}(h)$ is not contained in $\cup_{\ell \succ h} \text{ch}(\ell)$ for all $h \in \mathcal{H}$.

Example



$$
\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array}
\begin{array}{c} \begin{array}{ccc} h_1 & h_2 & h_3 \end{array} \\ \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & 0 & * \\ * & * & * \\ * & * & * \\ 0 & * & * \end{pmatrix} \end{array}
\overset{\text{ZUTA}}{\rightsquigarrow}
\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array}
\begin{array}{c} \begin{array}{ccc} h_1 & h_3 & h_2 \end{array} \\ \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & 0 \\ * & * & * \\ * & * & * \\ 0 & * & * \end{pmatrix} \end{array}
\underset{\text{upper-diag}\neq 0}{\overset{\text{upper-tri}=0}{\rightsquigarrow}}
\begin{array}{c} \\ v_1 \\ v_3 \\ v_2 \\ v_4 \\ v_5 \\ v_6 \end{array}
\begin{array}{c} \begin{array}{ccc} h_1 & h_3 & h_2 \end{array} \\ \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \\ * & * & * \\ * & * & * \\ 0 & * & * \end{pmatrix} \end{array}
$$

ZUTA $=$ "can permute cols and rows such that the upper right triangle of Λ is zero" and w.l.o.g. "diagonal entries are nonzero".

ZUTA and Generic Sign-identifiability

Recall: Literature centers around uniquely identifying Ω.

Lemma

Suppose G satisfies ZUTA and Ω is (generically) uniquely identifiable.

Then G is generically sign-identifiable.

Proof.

Take $(\widetilde{\Lambda}, \widetilde{\Omega}) \in \mathcal{F}_G(\Lambda, \Omega)$ for generically chosen (Λ, Ω). $\rightsquigarrow$ $\widetilde{\Lambda}\widetilde{\Lambda}^\top + \widetilde{\Omega} = \Lambda\Lambda^\top + \Omega$.

$\Longrightarrow$ $\widetilde{\Omega} = \Omega$, by assumption.

$\Longrightarrow$ $\widetilde{\Lambda}\widetilde{\Lambda}^\top = \Lambda\Lambda^\top$.

$\Longrightarrow$ $\widetilde{\Lambda} = \Lambda\Psi$ for $\Psi \in \{\pm 1\}^{|\mathcal{H}| \times |\mathcal{H}|}$ diagonal

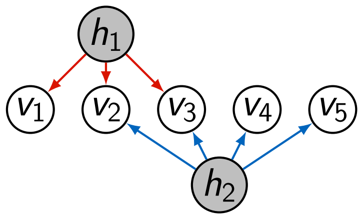
by ZUTA and uniqueness of the Cholesky decomposition. □

Anderson-Rubin Criterion

Theorem [Anderson, Rubin (1956)]

A factor analysis graph $G = (V \cup \mathcal{H}, D)$ that satisfies ZUTA is generically sign-identifiable if for any deleted row of the symbolic matrix $\Lambda = (\lambda_{vh}) \in \mathbb{R}^D$ there exist two disjoint submatrices that are generically of rank $|\mathcal{H}|$.

Example



$$\Lambda = \begin{pmatrix} \lambda_{v_1 h_1} & 0 \\ \lambda_{v_2 h_1} & \lambda_{v_2 h_2} \\ \lambda_{v_3 h_1} & \lambda_{v_3 h_2} \\ 0 & \lambda_{v_4 h_2} \\ 0 & \lambda_{v_5 h_2} \end{pmatrix}$$

<u>Observation:</u>

Need $|V| \geq 2|\mathcal{H}| + 1$.

Examples for Inconclusiveness of Anderson-Rubin

1)

$$\begin{pmatrix} \lambda_{v_1h_1} & 0 & 0 \\ \lambda_{v_2h_1} & \lambda_{v_2h_2} & 0 \\ 0 & \lambda_{v_3h_2} & \lambda_{v_3h_3} \\ \lambda_{v_4h_1} & 0 & \lambda_{v_4h_3} \\ 0 & \lambda_{v_5h_2} & 0 \\ 0 & 0 & \lambda_{v_6h_3} \end{pmatrix}$$

[Hosszejni, Frühwirth-Schnatter (2022)]

2)

$$\begin{pmatrix} \lambda_{v_1h_1} & 0 & 0 \\ \lambda_{v_2h_1} & \lambda_{v_2h_2} & 0 \\ \lambda_{v_3h_1} & \lambda_{v_3h_2} & \lambda_{v_3h_3} \\ \lambda_{v_4h_1} & \lambda_{v_4h_2} & \lambda_{v_4h_3} \\ \lambda_{v_5h_1} & \lambda_{v_5h_2} & \lambda_{v_5h_3} \\ \lambda_{v_6h_1} & \lambda_{v_6h_2} & 0 \end{pmatrix}$$

3)

$$\begin{pmatrix} \lambda_{v_1h_1} & 0 & 0 & 0 \\ \lambda_{v_2h_1} & \lambda_{v_2h_2} & 0 & 0 \\ 0 & \lambda_{v_3h_2} & \lambda_{v_3h_3} & 0 \\ \lambda_{v_4h_1} & 0 & \lambda_{v_4h_3} & \lambda_{v_4h_4} \\ 0 & \lambda_{v_5h_2} & 0 & 0 \\ 0 & 0 & \lambda_{v_6h_3} & 0 \\ 0 & 0 & 0 & \lambda_{v_7h_4} \\ 0 & 0 & 0 & \lambda_{v_8h_4} \\ 0 & 0 & 0 & \lambda_{v_9h_4} \end{pmatrix}$$

Investigation of Anderson-Rubin

1) Anderson-Rubin: Fix $v \in V$. Find disjoint $U, W \subseteq V \setminus \{v\}$ with $|U| = |W| = |\mathcal{H}|$ such that

$$\det(\Lambda_{U,\mathcal{H}}) \neq 0 \quad \text{and} \quad \det(\Lambda_{W,\mathcal{H}}) \neq 0 \quad \text{(not the zero polynomial)}$$
$$\iff \det([\Lambda\Lambda^\top]_{U,W}) \neq 0.$$

2) Consider the matrix with exactly one diagonal entry:

$$[\Lambda\Lambda^\top]_{\{v\}\cup U,\{v\}\cup W} = \left(\begin{array}{c|c} [\Lambda\Lambda^\top]_{vv} & [\Lambda\Lambda^\top]_{v,W} \\ \hline [\Lambda\Lambda^\top]_{U,v} & [\Lambda\Lambda^\top]_{W,U} \end{array}\right) = \left(\begin{array}{c|c} [\Lambda\Lambda^\top]_{vv} & \Sigma_{v,W} \\ \hline \Sigma_{U,v} & \Sigma_{U,W} \end{array}\right).$$

3) Solve for diagonal entry $[\Lambda\Lambda^\top]_{vv}$ by Laplace expansion:

$$0 = \det([\Lambda\Lambda^\top]_{\{v\}\cup U,\{v\}\cup W}) = [\Lambda\Lambda^\top]_{vv} \underbrace{\det(\Sigma_{U,W})}_{\neq 0} - \sum_{w\in W} \operatorname{sign}(w)\sigma_{vw} \det(\Sigma_{U,\{v\}\cup W\setminus\{w\}}).$$

4) Conclude: Solving for diagonal entries of $\Lambda\Lambda^\top$ is equivalent to solving for Ω.

$\overset{\text{ZUTA}}{\Longrightarrow}$ generic sign-identifiability.

Main Ideas for Sparse Setup

Exploit ZUTA: Operate column wise according to ZUTA-ordering $\prec$: $(h \prec \ell \prec \dots)$

1) Solve for first diagonal entry: $[\Lambda\Lambda^\top]_{v_h v_h} = \lambda_{v_h h}^2$.

   Solve for first column: $\frac{\sigma_{v_h w}}{\pm\lambda_{v_h h}} = \frac{\lambda_{v_h h}\lambda_{wh}}{\pm\lambda_{v_h h}} = \pm\lambda_{wh}$ for all $w \in \mathrm{ch}(h)$.

2) Solve for second diagonal entry: $[\Lambda\Lambda^\top]_{v_\ell v_\ell} = \lambda_{v_\ell h}^2 + \lambda_{v_\ell \ell}^2$.

   Solve for second column: ...

3) ...

$$\begin{array}{c} \\ v_h \\ v_\ell \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccc} \text{h} & \ell & \dots \end{array} \\ \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \\ \vdots & \vdots & \vdots \end{pmatrix} \end{array}$$

Local approach: Can also choose U, W such that $|W| = |U| < |\mathcal{H}|$.

$\rightsquigarrow$ Ensure that $\det([\Lambda\Lambda^\top]_{\{v\}\cup U, \{v\}\cup W}) = 0$ and that $\det([\Lambda\Lambda^\top]_{U,W}) \neq 0$.

Characterization: When is $\det([\Lambda\Lambda^\top]_{A,B}) = 0$ if Λ is sparse?

$\rightsquigarrow$ Intersection-free matchings (trek separation).

Intersection-free Matchings

- Consider paths of the form
$$a \leftarrow h \rightarrow b \qquad \text{for } a, b \in V \text{ and } h \in \mathcal{H}.$$
Relevance: $[\Lambda\Lambda^\top]_{ab} = \sum_{h \in \mathrm{pa}(a) \cap \mathrm{pa}(b)} \lambda_{ah}\lambda_{bh}$.

- System of paths $\Pi = \{\pi_1, \ldots, \pi_k\}$ is matching of $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$ if
$$\pi_i = a_i \leftarrow h_i \rightarrow b_i.$$

- A matching is intersection-free if all latent nodes h_i are distinct.

Lemma

For two subsets $A, B \subseteq V$ with $|A| = |B|$ it holds that $\det([\Lambda\Lambda^\top]_{A,B}) \neq 0$ if and only if there is an intersection-free matching of A and B.

Application of trek separation by Sullivant, Talaska and Draisma (2010).

Matching Criterion

Definition

Fix a latent node $h \in \mathcal{H}$. Tuple $(v, U, W, S) \in V \times 2^V \times 2^V \times 2^{\mathcal{H}\setminus\{h\}}$ satisfies the matching criterion for h if

(i) $\mathrm{pa}(v) \setminus S = \{h\}$ and $v \notin U \cup W$,

(ii) U and W are disjoint, nonempty sets of equal cardinality,

(iii) there exists an intersection-free matching of U and W that avoids S,

(iv) there does not exist an intersection-free matching of $\{v\} \cup W$ and $\{v\} \cup U$ that avoids S.

$S =$ "solved nodes".

By (iii), $\det([\Lambda\Lambda^\top]_{U,W}) \neq 0$.

By (iv), $\det([\Lambda\Lambda^\top]_{\{v\}\cup U,\{v\}\cup W}) = 0$.

Algorithm: Recursive Solving

Theorem (M-identifiability)

If the tuple (v, U, W, S) satisfies the matching criterion with respect to h and all nodes $\ell \in S$ are "solved before", then we can "solve" for h.

That is, $(\widetilde{\Omega}, \widetilde{\Lambda}) \in \mathcal{F}_G(\Omega, \Lambda) \implies \widetilde{\Lambda}_{\mathrm{ch}(h),h} = \pm\Lambda_{\mathrm{ch}(h),h}$.

Algorithm (`M-ID`)

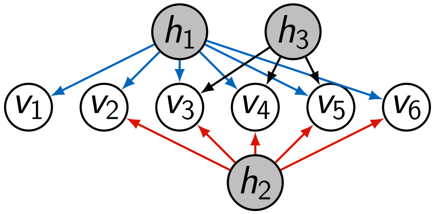
- Cycle through latent nodes h and search for tuples (v, U, W, S).
- Network-flow setup finds suitable tuples in polynomial time under a bound on $|U| = |W|$.

  Conjecture: *If we do not bound the cardinality $|U| = |W|$, then `M-ID` is NP-complete.*

Remarks

- Subsumes Anderson-Rubin.
- Together with an extension, subsumes anything we know (e.g. Bekker and ten Berge, 1997).

Example 1



$$\Lambda = \begin{pmatrix} \lambda_{v_1 h_1} & 0 & 0 \\ \lambda_{v_2 h_1} & \lambda_{v_2 h_2} & 0 \\ \lambda_{v_3 h_1} & \lambda_{v_3 h_2} & \lambda_{v_3 h_3} \\ \lambda_{v_4 h_1} & \lambda_{v_4 h_2} & \lambda_{v_4 h_3} \\ \lambda_{v_5 h_1} & \lambda_{v_5 h_2} & \lambda_{v_5 h_3} \\ \lambda_{v_6 h_1} & \lambda_{v_6 h_2} & 0 \end{pmatrix}$$

h_1: Take $v = v_1$, $S = \emptyset$ and $U = \{v_2, v_6\}$, $W = \{v_3, v_4\}$.

(iii) $v_2 \leftarrow h_1 \rightarrow v_3$, $v_6 \leftarrow h_2 \rightarrow v_4$; (iv) $\text{pa}(\{v\} \cup U) \cap \text{pa}(\{v\} \cup W) = \{h_1, h_2\}$.

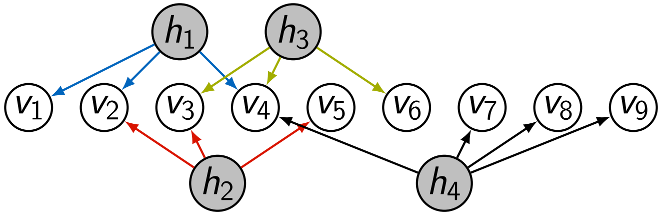
h_2: Take $v = v_2$, $S = \{h_1\}$ and $U = \{v_3\}$, $W = \{v_6\}$.

(iii) $v_3 \leftarrow h_2 \rightarrow v_6$; (iv) $(\text{pa}(\{v\} \cup U) \cap \text{pa}(\{v\} \cup W)) \setminus S = \{h_2\}$.

h_3: Take $v = v_3$, $S = \{h_1, h_2\}$ and $U = \{v_4\}$, $W = \{v_5\}$.

(iii) $v_4 \leftarrow h_3 \rightarrow v_5$; (iv) $(\text{pa}(\{v\} \cup U) \cap \text{pa}(\{v\} \cup W)) \setminus S = \{h_3\}$.

Example 2



$$\Lambda = \begin{pmatrix} \lambda_{v_1h_1} & 0 & 0 & 0 \\ \lambda_{v_2h_1} & \lambda_{v_2h_2} & 0 & 0 \\ 0 & \lambda_{v_3h_2} & \lambda_{v_3h_3} & 0 \\ \lambda_{v_4h_1} & 0 & \lambda_{v_4h_3} & \lambda_{v_4h_4} \\ 0 & \lambda_{v_5h_2} & 0 & 0 \\ 0 & 0 & \lambda_{v_6h_3} & 0 \\ 0 & 0 & 0 & \lambda_{v_7h_4} \\ 0 & 0 & 0 & \lambda_{v_8h_4} \\ 0 & 0 & 0 & \lambda_{v_9h_4} \end{pmatrix}$$

h_1: Take $v = v_1$, $S = \emptyset$ and $U = \{v_2\}$, $W = \{v_4\}$.

(iii) $v_2 \leftarrow h_1 \rightarrow v_4$; (iv) $\text{pa}(\{v\} \cup U) \cap \text{pa}(\{v\} \cup W) = \{h_1\}$.

h_2: Take $v = v_2$, $S = \{h_1\}$ and $U = \{v_3\}$, $W = \{v_5\}$.

(iii) $v_3 \leftarrow h_2 \rightarrow v_5$; (iv) $(\text{pa}(\{v\} \cup U) \cap \text{pa}(\{v\} \cup W)) \setminus S = \{h_2\}$.

h_3: Take $v = v_3$, $S = \{h_1, h_2\}$ and $U = \{v_4\}$, $W = \{v_6\}$.

(iii) $v_4 \leftarrow h_3 \rightarrow v_6$; (iv) $(\text{pa}(\{v\} \cup U) \cap \text{pa}(\{v\} \cup W)) \setminus S = \{h_3\}$.

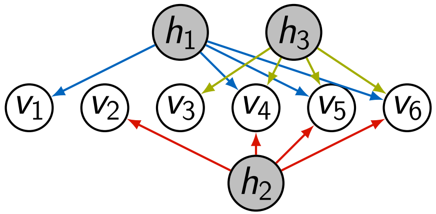
h_4: Take $v = v_4$, $S = \{h_1, h_2, h_3\}$ and $U = \{v_7\}$, $W = \{v_8\}$.

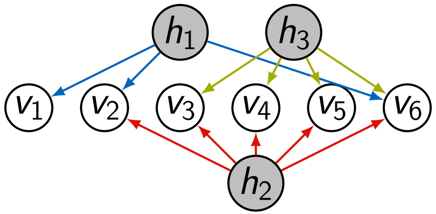
(iii) $v_7 \leftarrow h_4 \rightarrow v_8$; (iv) $(\text{pa}(\{v\} \cup U) \cap \text{pa}(\{v\} \cup W)) \setminus S = \{h_4\}$.

Numerical Experiments

Experiment 1: Counts of unlabeled sparse factor graphs satisfying ZUTA with at most $|V| = 7$ observed nodes and $|\mathcal{H}| = 3$ latent nodes.

ZUTA	gen. sign-id	AR-id	M-id	Ext. M-id
562	174	150	172	172





Experiment 2: Counts of unlabeled sparse factor graphs with at most $|V| = 9$ observed nodes and $|\mathcal{H}| = 4$ latent nodes.

total	ZUTA	AR-id	M-id	Ext. M-id
64166	46260	16104	20951	21568

Conclusion

- Sparse factor analysis models are the "building block" for many latent variable models.
- As projections, latent variable models may feature complicated parametrizations and geometry.
- Even in "simple" factor analysis models, there is still lots to explore ...
- Identification results have applications also in exploratory factor analysis.

Articles:

Sturma, Kranzlmueller, Portakal, Drton (2026).
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SIAM Journal on Applied Algebra and Geometry, Vol. 9, No. 2, 279-309.