Identifiability in Sparse Factor Analysis

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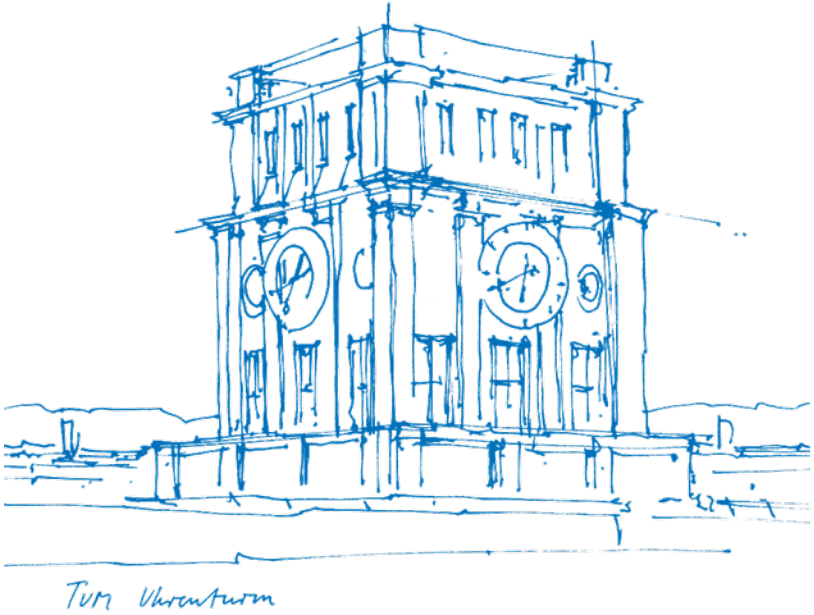
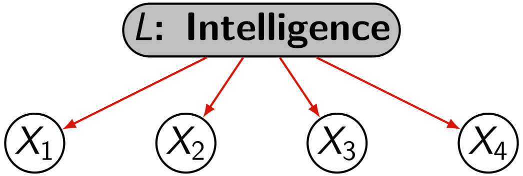


Spearman's One-Factor Model (1904)



Linear structural equations:

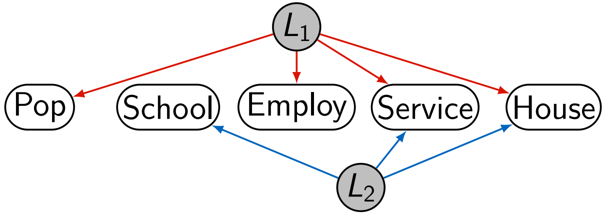
$$\begin{aligned} X_1 &= \lambda_{10} + \lambda_{1L} L + \varepsilon_1, \\ X_2 &= \lambda_{20} + \lambda_{2L} L + \varepsilon_2, \\ X_3 &= \lambda_{30} + \lambda_{3L} L + \varepsilon_3, \\ X_4 &= \lambda_{40} + \lambda_{4L} L + \varepsilon_4. \end{aligned}$$

Jointly independent errors: $\varepsilon_1, \dots, \varepsilon_4$.

$\mathrm{Var}[\varepsilon_v] = \omega_{vv} < \infty$, $\mathrm{Var}[L] = 1$.

Topic: Can we recover the "factor loadings" λ_{vL} and the "error variances" ω_{vv} from $\Sigma = \mathrm{Var}[X]$?

Socio-economic Example from Harmann (1976), Modern Factor Analysis



$$X = \Lambda L + \varepsilon, \quad \text{where } \Lambda = \begin{pmatrix} \lambda_{11} & 0 \\ 0 & \lambda_{22} \\ \lambda_{31} & 0 \\ \lambda_{41} & \lambda_{42} \\ \lambda_{51} & \lambda_{52} \end{pmatrix}.$$

Observed covariance matrix (when $\mathrm{Var}[L] = I$):

$$\Sigma = \Lambda\Lambda^\top + \Omega = \begin{pmatrix} \omega_{11} + \lambda_{11}^2 & 0 & \boxed{\lambda_{11}\lambda_{31}} & \boxed{\lambda_{11}\lambda_{41}} & \lambda_{11}\lambda_{51} \\ 0 & \omega_{22} + \lambda_{22}^2 & 0 & \lambda_{22}\lambda_{42} & \lambda_{22}\lambda_{52} \\ \lambda_{11}\lambda_{31} & 0 & \omega_{33} + \lambda_{31}^2 & \boxed{\lambda_{31}\lambda_{41}} & \lambda_{31}\lambda_{51} \\ \lambda_{11}\lambda_{41} & \lambda_{22}\lambda_{42} & \lambda_{31}\lambda_{41} & \omega_{44} + \lambda_{41}^2 + \lambda_{52}^2 & \lambda_{41}\lambda_{51} + \lambda_{42}\lambda_{52} \\ \lambda_{11}\lambda_{51} & \lambda_{22}\lambda_{52} & \lambda_{31}\lambda_{51} & \lambda_{41}\lambda_{51} + \lambda_{42}\lambda_{52} & \omega_{55} + \lambda_{51}^2 + \lambda_{52}^2 \end{pmatrix}.$$

We see that

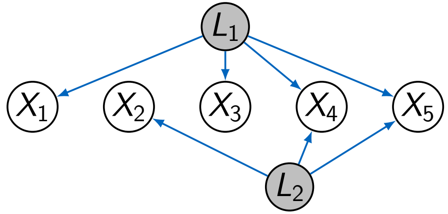
1) $\sqrt{\dfrac{\sigma_{13}\sigma_{14}}{\sigma_{34}}} = \sqrt{\dfrac{\lambda_{11}\lambda_{31}\,\lambda_{11}\lambda_{41}}{\lambda_{31}\lambda_{41}}} = \sqrt{\lambda_{11}^2} = a_1\lambda_{11}$ with $a_1 \in \{\pm 1\}$ and $\sigma_{34} = \lambda_{31}\lambda_{41} \neq 0$ 'almost surely',

2) $\dfrac{\sigma_{13}}{\sqrt{\sigma_{13}\sigma_{14}/\sigma_{34}}} = \dfrac{\lambda_{11}\lambda_{31}}{a_1\lambda_{11}} = a_1\lambda_{31}$ with $\lambda_{11} \neq 0$ 'almost surely'.

$\Longrightarrow$ Can identify $\Lambda_{\mathrm{ch}(L_1),L_1}$ up to column-sign, similarly $\Lambda_{\mathrm{ch}(L_2),L_2}$.

Factor Analysis vs. Causal Representation Learning

Sparse Factor Analysis

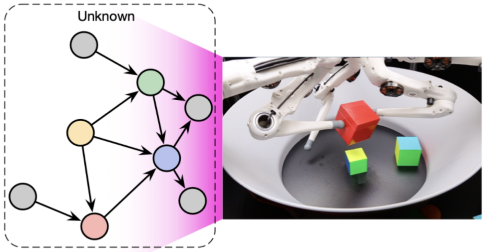


Model:

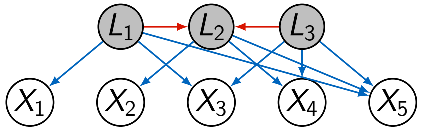
$$X = \Lambda L + \varepsilon,$$

where Λ sparse.

Causal Representation Learning



[Schölkopf et al. (2021)]



Model:

$$X = GL + \varepsilon_X,$$
$$L = ML + \varepsilon_L,$$

where G and M are sparse.

$$\Longrightarrow \quad X = \underbrace{G(I - M)^{-1}}_{=:\Lambda} \varepsilon_L + \varepsilon_X$$

Understanding sparse factor analysis is key for causal representation learning!

Setup

Variables:

Observed: $X = (X_v)_{v \in V}$ Latent: $L = (L_h)_{h \in \mathcal{H}}$

Graph:

Bipartite directed graph $G = (V \dot{\cup} \mathcal{H}, D)$, that is, $D \subseteq \mathcal{H} \times V$.

Sparse factor analysis model:

$$X = \Lambda L + \varepsilon$$

– all latent factors and error terms in (L, ε) are mutually independent, so $\Omega = \text{Var}[\varepsilon] = \text{diag}(\omega_{vv} : v \in V)$ diagonal, and $\text{Var}[L] = I$.

– parameter matrix Λ is sparse and supported over edge set D (write $\Lambda \in \mathbb{R}^D$).

Content of the Talk

Definition

Every factor analysis graph G yields a parametrization of the observed covariance matrix:

$$\tau_G : (\Lambda, \Omega) \longmapsto \Sigma \equiv \Lambda\Lambda^\top + \Omega.$$

Fiber: $\mathcal{F}_G(\Omega, \Lambda) = \{(\widetilde{\Omega}, \tilde{\Lambda}) : \tau_G(\widetilde{\Omega}, \tilde{\Lambda}) = \tau_G(\Omega, \Lambda)\}.$

The model given by G is generically sign-identifiable if

$$\mathcal{F}_G(\Omega, \Lambda) = \{(\widetilde{\Omega}, \tilde{\Lambda}) : \widetilde{\Omega} = \Omega \text{ and } \tilde{\Lambda} = \Lambda\Psi \text{ for } \Psi \in \{\pm 1\}^{|\mathcal{H}|\times|\mathcal{H}|} \text{ diagonal}\} \qquad \text{for 'almost all' } (\Lambda, \Omega).$$

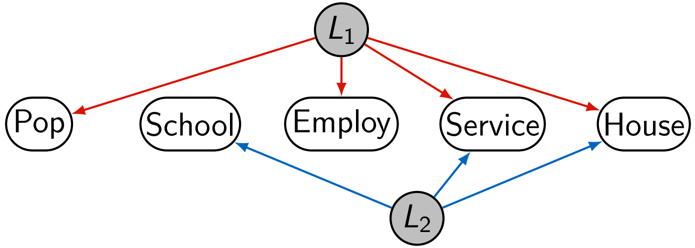
Main Contribution:

- Sufficient graphical condition for generic sign-identifiability.
- Recursive polynomial time algorithm.

  (caveat: polynomial time when bounding a cardinality in a search step)

Gröbner basis computations solve the problem ... on a small scale.

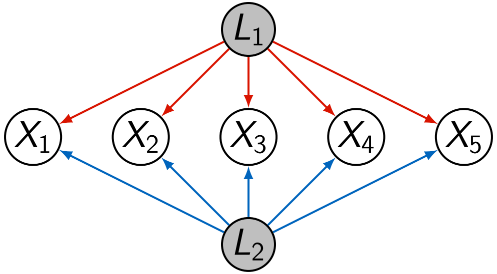
Our Software




```
# Define graph
> L = matrix(c(1, 0,
+              0, 1,
+              1, 0,
+              1, 1,
+              1, 1), 5, 2, byrow=TRUE)
> g = FactorGraph(L)
>
> # Check identifiability
> res = mID(g)
Generic Sign-Identifiability Summary
# nr. of latent nodes that are gen. sign-identifiable: 2
# gen. sign-identifiable nodes: 1, 2
```

Available at `https://github.com/MiriamKranzlmueller/id-factor-analysis`.

Rotational Indeterminacy



- No restriction on Λ:

$$\Lambda\Lambda^\top + \Omega = (\Lambda Q)(Q^\top \Lambda^\top) + \Omega \quad \text{for all } Q \text{ orthogonal.}$$

- Literature: Focus on identifiability of Ω and $\Lambda\Lambda^\top$ in full factor analysis models.

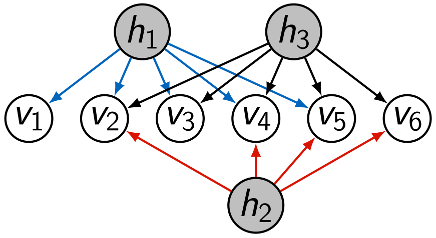
- However, if Λ is sparse, we might get generic sign-identifiability.

Zero Upper Triangular Assumption

Definition

The graph G satisfies the Zero Upper Triangular Assumption (ZUTA) if there exists an ordering $\prec$ on the latent nodes $\mathcal{H}$ such that $\mathrm{ch}(h)$ is not contained in $\cup_{\ell \succ h} \mathrm{ch}(\ell)$ for all $h \in \mathcal{H}$.

Example



$$
\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array}
\begin{array}{c} \begin{matrix} h_1 & h_2 & h_3 \end{matrix} \\ \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & 0 & * \\ * & * & * \\ * & * & * \\ 0 & * & * \end{pmatrix} \end{array}
\overset{\text{ZUTA}}{\rightsquigarrow}
\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array}
\begin{array}{c} \begin{matrix} h_1 & \color{orange}{h_3} & \color{orange}{h_2} \end{matrix} \\ \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & 0 \\ * & * & * \\ * & * & * \\ 0 & * & * \end{pmatrix} \end{array}
\underset{\text{upper-diag}\neq 0}{\overset{\text{upper-tri}=0}{\rightsquigarrow}}
\begin{array}{c} \\ v_1 \\ \color{orange}{v_3} \\ \color{orange}{v_2} \\ v_4 \\ v_5 \\ v_6 \end{array}
\begin{array}{c} \begin{matrix} h_1 & h_3 & h_2 \end{matrix} \\ \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \\ * & * & * \\ * & * & * \\ 0 & * & * \end{pmatrix} \end{array}
$$

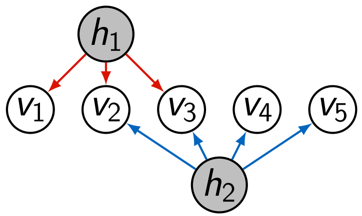
ZUTA $=$ "can permute cols and rows such that the upper right triangle of Λ is zero"
and w.l.o.g. "diagonal entries are nonzero".

Anderson-Rubin Criterion

Theorem [Anderson, Rubin (1956)]

A factor analysis graph $G = (V \cup \mathcal{H}, D)$ that satisfies ZUTA is generically sign-identifiable if for any deleted row of the symbolic matrix $\Lambda = (\lambda_{vh}) \in \mathbb{R}^D$ there exist two disjoint submatrices that are generically of rank $|\mathcal{H}|$.

Example



$$\Lambda = \begin{pmatrix} \lambda_{v_1 h_1} & 0 \\ \lambda_{v_2 h_1} & \lambda_{v_2 h_2} \\ \lambda_{v_3 h_1} & \lambda_{v_3 h_2} \\ 0 & \lambda_{v_4 h_2} \\ 0 & \lambda_{v_5 h_2} \end{pmatrix}$$

<u>Observation:</u>

Need $|V| \geq 2|\mathcal{H}| + 1$.

Examples for Inconclusiveness of Anderson-Rubin

1)

$$\begin{pmatrix} \lambda_{v_1h_1} & 0 & 0 \\ \lambda_{v_2h_1} & \lambda_{v_2h_2} & 0 \\ 0 & \lambda_{v_3h_2} & \lambda_{v_3h_3} \\ \lambda_{v_4h_1} & 0 & \lambda_{v_4h_3} \\ 0 & \lambda_{v_5h_2} & 0 \\ 0 & 0 & \lambda_{v_6h_3} \end{pmatrix}$$

[Hosszejni, Frühwirth-Schnatter (2022)]

2)

$$\begin{pmatrix} \lambda_{v_1h_1} & 0 & 0 \\ \lambda_{v_2h_1} & \lambda_{v_2h_2} & 0 \\ \lambda_{v_3h_1} & \lambda_{v_3h_2} & \lambda_{v_3h_3} \\ \lambda_{v_4h_1} & \lambda_{v_4h_2} & \lambda_{v_4h_3} \\ \lambda_{v_5h_1} & \lambda_{v_5h_2} & \lambda_{v_5h_3} \\ \lambda_{v_6h_1} & \lambda_{v_6h_2} & 0 \end{pmatrix}$$

3)

$$\begin{pmatrix} \lambda_{v_1h_1} & 0 & 0 & 0 \\ \lambda_{v_2h_1} & \lambda_{v_2h_2} & 0 & 0 \\ 0 & \lambda_{v_3h_2} & \lambda_{v_3h_3} & 0 \\ \lambda_{v_4h_1} & 0 & \lambda_{v_4h_3} & \lambda_{v_4h_4} \\ 0 & \lambda_{v_5h_2} & 0 & 0 \\ 0 & 0 & \lambda_{v_6h_3} & 0 \\ 0 & 0 & 0 & \lambda_{v_7h_4} \\ 0 & 0 & 0 & \lambda_{v_8h_4} \\ 0 & 0 & 0 & \lambda_{v_9h_4} \end{pmatrix}$$

Investigation of Anderson-Rubin

1) Anderson-Rubin: Fix $v \in V$. Find disjoint $U, W \subseteq V \setminus \{v\}$ with $|U| = |W| = |\mathcal{H}|$ such that

$$\det(\Lambda_{U,\mathcal{H}}) \neq 0 \quad \text{and} \quad \det(\Lambda_{W,\mathcal{H}}) \neq 0 \quad \text{(not the zero polynomial)}$$
$$\iff \det([\Lambda\Lambda^\top]_{U,W}) \neq 0.$$

2) Consider the matrix with exactly one diagonal entry:

$$[\Lambda\Lambda^\top]_{\{v\}\cup U,\{v\}\cup W} = \left(\begin{array}{c|c} [\Lambda\Lambda^\top]_{vv} & [\Lambda\Lambda^\top]_{v,W} \\ \hline [\Lambda\Lambda^\top]_{U,v} & [\Lambda\Lambda^\top]_{W,U} \end{array}\right) = \left(\begin{array}{c|c} [\Lambda\Lambda^\top]_{vv} & \Sigma_{v,W} \\ \hline \Sigma_{U,v} & \Sigma_{U,W} \end{array}\right).$$

3) Solve for diagonal entry $[\Lambda\Lambda^\top]_{vv}$ by Laplace expansion:

$$0 = \det([\Lambda\Lambda^\top]_{\{v\}\cup U,\{v\}\cup W}) = [\Lambda\Lambda^\top]_{vv} \underbrace{\det(\Sigma_{U,W})}_{\neq 0} - \sum_{w\in W} \text{sign}(w)\sigma_{vw} \det(\Sigma_{U,\{v\}\cup W\setminus\{w\}}).$$

4) Conclude: Solving for diagonal entries of $\Lambda\Lambda^\top$ is equivalent to solving for Ω.

$\implies$ Equivalent to solving for $\Sigma - \Omega = \Lambda\Lambda^\top$.

$\overset{\text{ZUTA}}{\implies}$ generic sign-identifiability (uniqueness of Cholesky decomposition).

Main Idea for Sparse Setup

Local approach: Can also choose U, W such that $|W| = |U| < |\mathcal{H}|$.

$\rightsquigarrow$ Ensure that $\det([\Lambda\Lambda^\top]_{U,W}) \neq 0$ and that $\det([\Lambda\Lambda^\top]_{\{v\}\cup U,\{v\}\cup W}) = 0$.

Characterization: When is $\det([\Lambda\Lambda^\top]_{A,B}) = 0$ if Λ is sparse?

$\rightsquigarrow$ Intersection-free matchings.

Definition

System of paths $\Pi = \{\pi_1, \dots, \pi_k\}$ is matching of $A = \{a_1, \dots, a_k\} \subseteq V$ and $B = \{b_1, \dots, b_k\} \subseteq V$ if

$$\pi_i = a_i \leftarrow h_i \rightarrow b_i.$$

A matching is intersection-free if all latent nodes h_i are distinct.

Lemma

For two subsets $A, B \subseteq V$ with $|A| = |B|$ it holds that $\det([\Lambda\Lambda^\top]_{A,B}) \neq 0$ if and only if there is an intersection-free matching of A and B.

Matching Criterion

Definition

Fix a latent node $h \in \mathcal{H}$. Tuple $(v, U, W, S) \in V \times 2^V \times 2^V \times 2^{\mathcal{H}\setminus\{h\}}$ satisfies the matching criterion for h if

(i) $\mathrm{pa}(v) \setminus S = \{h\}$ and $v \notin U \cup W$,

(ii) U and W are disjoint, nonempty sets of equal cardinality,

(iii) there exists an intersection-free matching of U and W that avoids S,

(iv) there does not exist an intersection-free matching of $\{v\} \cup W$ and $\{v\} \cup U$ that avoids S.

S = "solved nodes".

By (iii), $\det([\Lambda\Lambda^\top]_{U,W}) \neq 0$.

By (iv), $\det([\Lambda\Lambda^\top]_{\{v\}\cup U,\{v\}\cup W}) = 0$.

Algorithm: Recursive Solving

Theorem (M-identifiability)

If the tuple (v, U, W, S) satisfies the matching criterion with respect to h <u>*and*</u> *all nodes $\ell \in S$ are "solved before", then we can "solve" for h.*

That is, $(\widetilde{\Omega}, \widetilde{\Lambda}) \in \mathcal{F}_G(\Omega, \Lambda) \implies \widetilde{\Lambda}_{\mathrm{ch}(h),h} = \pm\Lambda_{\mathrm{ch}(h),h}$.

Algorithm (`M-ID`)

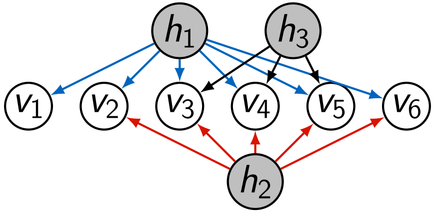
- Cycle through latent nodes h and search for tuples (v, U, W, S).
- Network-flow setup finds suitable tuples in polynomial time under a bound on $|U| = |W|$.

  Conjecture: *If we do not bound the cardinality $|U| = |W|$, then `M-ID` is NP-complete.*

Remarks

- Subsumes Anderson-Rubin.
- Together with an extension, subsumes anything we know (e.g. Bekker and ten Berge, 1997).

Example



$$\Lambda = \begin{pmatrix} \lambda_{v_1h_1} & 0 & 0 \\ \lambda_{v_2h_1} & \lambda_{v_2h_2} & 0 \\ \lambda_{v_3h_1} & \lambda_{v_3h_2} & \lambda_{v_3h_3} \\ \lambda_{v_4h_1} & \lambda_{v_4h_2} & \lambda_{v_4h_3} \\ \lambda_{v_5h_1} & \lambda_{v_5h_2} & \lambda_{v_5h_3} \\ \lambda_{v_6h_1} & \lambda_{v_6h_2} & 0 \end{pmatrix}$$

h_1: Take $v = v_1$, $S = \emptyset$ and $U = \{v_2, v_6\}$, $W = \{v_3, v_4\}$.

(iii) $v_2 \leftarrow h_1 \rightarrow v_3$, $v_6 \leftarrow h_2 \rightarrow v_4$; (iv) $\mathrm{pa}(\{v\} \cup U) \cap \mathrm{pa}(\{v\} \cup W) = \{h_1, h_2\}$.

h_2: Take $v = v_2$, $S = \{h_1\}$ and $U = \{v_3\}$, $W = \{v_6\}$.

(iii) $v_3 \leftarrow h_2 \rightarrow v_6$; (iv) $(\mathrm{pa}(\{v\} \cup U) \cap \mathrm{pa}(\{v\} \cup W)) \setminus S = \{h_2\}$.

h_3: Take $v = v_3$, $S = \{h_1, h_2\}$ and $U = \{v_4\}$, $W = \{v_5\}$.

(iii) $v_4 \leftarrow h_3 \rightarrow v_5$; (iv) $(\mathrm{pa}(\{v\} \cup U) \cap \mathrm{pa}(\{v\} \cup W)) \setminus S = \{h_3\}$.

Conclusion

- Sparse factor analysis models are the "building block" for many latent variable models.
- Latent variable models generally feature complicated parametrizations.
- Even in "simple" factor analysis models, there is still lots to explore ...
- Papers:

Sturma, Kranzlmueller, Portakal, Drton (2025).
Matching Criterion for Identifiability in Sparse Factor Analysis.
arXiv preprint arXiv:2502.02986.

Drton, Grosdos, Portakal, Sturma (2023).
Algebraic Sparse Factor Analysis.
arXiv preprint arXiv:2312.14762.

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