EPFL

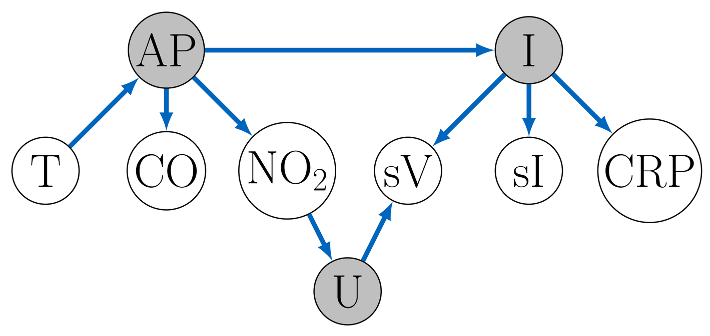
Trek-Based Parameter Identification for Linear Causal Models With Arbitrarily Structured Latent Variables

Nils Sturma and Mathias Drton

École Polytechnique Fédérale de Lausanne and Technical University of Munich

Motivation & Setup

Directed graph captures causal relations between variables. Example:



Setup:

- Directed graph $G = (V, D)$ with observed and latent nodes $V = \mathcal{O} \sqcup \mathcal{L}$.
- Random vector $X = (X_v)_{v \in V}$ satisfies

$$X_v = \sum_{w \in \mathrm{pa}(v)} \lambda_{wv} X_w + \varepsilon_v,$$

  where $\varepsilon = (\varepsilon_v)_{v \in V}$ are jointly independent with $\mathrm{Var}[\varepsilon_v] = \phi_v$.
- Matrix form: $X = \Lambda^\top X + \varepsilon \quad \Longrightarrow \quad \Sigma := \mathrm{Cov}[X] = (I - \Lambda)^{-\top} \Phi (I - \Lambda)^{-1}$.

Goal: Identify *semi-direct effects* between observed variables from $\mathrm{Cov}[X]$.

Semi-direct effect $v \rightsquigarrow w$ corresponds to sum over all paths of the form

$$v \to w \quad \text{or} \quad v \to h_1 \to \cdots \to h_k \to w \quad \text{with } h_i \in \mathcal{H} \text{ for all } i.$$

$\Longrightarrow$ Identify $\overline{\Lambda} := \Lambda_{\mathcal{O},\mathcal{O}} + \Lambda_{\mathcal{O},\mathcal{L}} (I - \Lambda_{\mathcal{L},\mathcal{L}})^{-1} \Lambda_{\mathcal{L},\mathcal{O}}$.

Main contributions:

- Sufficient graphical condition for rational identifiability of $\overline{\Lambda}$.
- Efficient sound and complete algorithm.
- Allows to transform model to simpler measurement model.

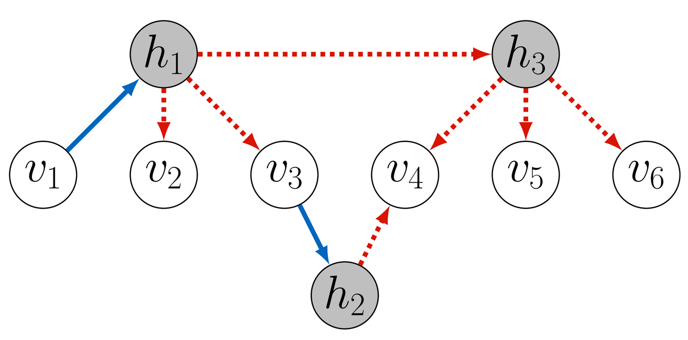
Factorization of the Covariance Matrix

$$\Sigma = (I - \overline{\Lambda})^{-\top} \Omega (I - \overline{\Lambda})^{-1}.$$

Observation:

Ω is covariance matrix in the model given by subgraph G_{lat} obtained from removing all edges $v \to w$ with $v \in \mathcal{O}$.

Example:



Key Idea

Fix $v \in \mathcal{O}$ and consider $\overline{\Lambda}_{\overline{\mathrm{pa}}(v),v}$. Find linear equation system

$$\left([(I - \overline{\Lambda})^\top \Sigma]_{Y,\overline{\mathrm{pa}}(v)} \quad \Omega_{Y,Z} \right) \cdot \begin{pmatrix} \overline{\Lambda}_{\overline{\mathrm{pa}}(v),v} \\ \psi \end{pmatrix} = [(I - \overline{\Lambda})^\top \Sigma]_{Y,v} \qquad (*)$$

such that the matrix on the left-hand side is *invertible* and that $\Omega_{Y,Z}$ and suitable entries in $\overline{\Lambda}$ are *already certified to be identifiable* from earlier calculations.

Requires new proof technique:

Trek rule: $\Sigma_{vw} = [(I - \Lambda)^{-\top} \Phi (I - \Lambda)^{-1}]_{vw}$ is sum over 'treks' $v \leftarrow \cdots \leftarrow k \rightarrow \cdots \rightarrow w$.

$$\Longrightarrow \Sigma_{vw} = \sum_{\pi \in \mathcal{T}(v,w)} \phi_k \prod_{x \to y \in \pi} \lambda_{xy}.$$

Similarly:

- Ω_{vw} is sum over treks in G_{lat}.
- $[(I - \overline{\Lambda})^\top \Sigma]_{vw} = [\Omega (I - \overline{\Lambda})^{-1}]_{vw}$ is sum over treks with left part in G_{lat}.

Trek-Separation in Subgraphs. If there exists a system of treks with no sided intersection from Y to $\overline{\mathrm{pa}}(v) \cup Z$ in G such that the left part of every trek only takes edges in G_{lat}, and the right part of every trek ending in Z only takes edges in G_{lat}, then we have generically

$$\det \left([(I - \overline{\Lambda})^\top \Sigma]_{Y,\overline{\mathrm{pa}}(v)} \quad \Omega_{Y,Z} \right) \neq 0.$$

Latent Subgraph Criterion

Definition. Given a node $v \in \mathcal{O}$, the 4-tuple $(Y, Z, H_1, H_2) \in 2^{\mathcal{O} \setminus \{v\}} \times 2^{\mathcal{O} \setminus \{v\}} \times 2^{\mathcal{L}} \times 2^{\mathcal{L}}$ satisfies the *latent-subgraph criterion* (LSC) for v if

(i) $|Y| = |\overline{\mathrm{pa}}(v)| + |Z|$ and $|Z| = |H_1| + |H_2|$ with $Z \cap \overline{\mathrm{pa}}(v) = \emptyset$,

(ii) $Y \cap (Z \cup \{v\}) = \emptyset$ and (H_1, H_2) trek separates Y and $Z \cup \{v\}$ in the latent subgraph G_{lat},

(iii) there exists a system of treks with no sided intersection from Y to $\overline{\mathrm{pa}}(v) \cup Z$ in G such that the left part of every trek only takes edges in G_{lat}, and the right part of every trek ending in Z only takes edges in G_{lat}.

Interpretation:

- (i) - (iii) ensure that there is $\psi \in \mathbb{R}^{|Z|}$ s.t. $\Omega_{Y,Z} \cdot \psi = \Omega_{Y,v}$.

$$\Longrightarrow [(I - \overline{\Lambda})^\top \Sigma (I - \overline{\Lambda})]_{Y,v} - \Omega_{Y,Z} \cdot \psi = 0.$$

$$\Longrightarrow (*) \text{ holds.}$$

- (iii) ensures invertability.

Theorem. If the tuple $(Y, Z, H_1, H_2) \in 2^{\mathcal{O} \setminus \{v\}} \times 2^{\mathcal{O} \setminus \{v\}} \times 2^{\mathcal{L}} \times 2^{\mathcal{L}}$ satisfies the LSC for $v \in \mathcal{O}$, then $\overline{\Lambda}_{*,v}$ is a a rational function of the observed covariance matrix Σ, the semi-direct effects $(\overline{\Lambda}_{*,z})_{z \in Z}$, and the semi-direct effects $(\overline{\Lambda}_{*,y})_{y \in Y}$ for those $y \in Y$ that can be reached from $Z \cup \{v\}$ via a trek whose left part only takes edges in G_{lat} and avoids H_2, H_1.

Software

Algorithm. Recursively cycle through nodes $v \in \mathcal{O}$ to find LSC-tuples solving for $\overline{\Lambda}_{*,v}$.

- Find LSC-tuples via integer linear program that generalizes classical max-flow problem.
- Sound and complete.
- GitHub: `NilsSturma/LSC`; soon on `CRAN` within `SEMID` package.

Example

1) The graph displayed on the left is LSC-identifiable.

<u>$v = v_1$</u>: $(Y, Z, H_1, H_2) = (\emptyset, \emptyset, \emptyset, \emptyset)$.

<u>$v = v_2, v_3, v_5, v_6$</u>: $(Y, Z, H_1, H_2) = (\{v_1\}, \emptyset, \emptyset, \emptyset)$ and $\Pi = \{v_1\}$.

<u>$v = v_4$</u>: $(Y, Z, H_1, H_2) = (\{v_1, v_2, v_3\}, \{v_5\}, \emptyset, \{h_1\})$ and

$$\Pi = \{v_1, \ v_3, \ v_2 \leftarrow h_1 \rightarrow h_2 \rightarrow v_5\}.$$

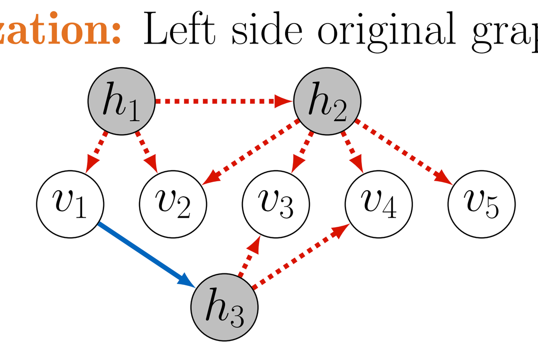
2) Now, consider the simpler model given by the subgraph G_{lat} with covariance matrix Ω. It is well known that $h_1 \to h_3$ is identifiable (up to sign) via

$$\sqrt{\frac{\omega_{v_2 v_6} \omega_{v_3 v_5}}{\omega_{v_2 v_3} \omega_{v_5 v_6} - \omega_{v_2 v_6} \omega_{v_3 v_5}}} = \sqrt{\lambda_{h_1 h_3}^2} = |\lambda_{h_1 h_3}|,$$

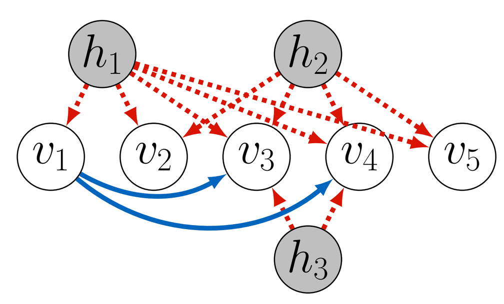
when fixing $\mathrm{Var}[\varepsilon_h] = \phi_h = 1$ for all $h \in \mathcal{L}$.

Subtleties

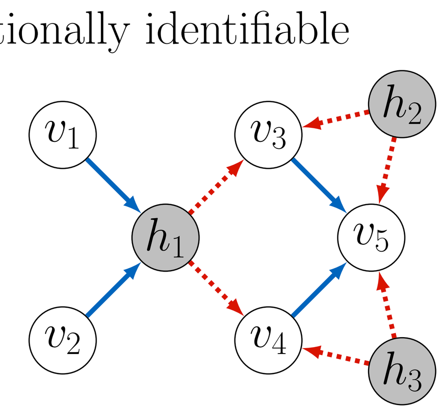
Canonicalization: Left side original graph, right side its canonicalization.



rationally identifiable



not rationally identifiable



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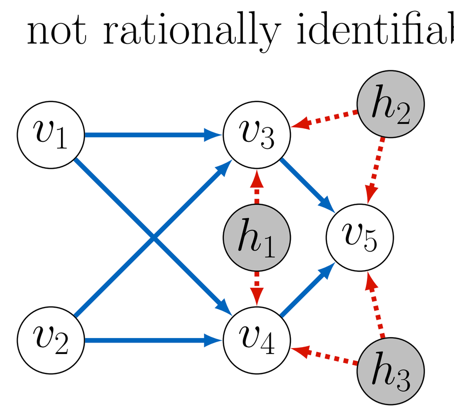
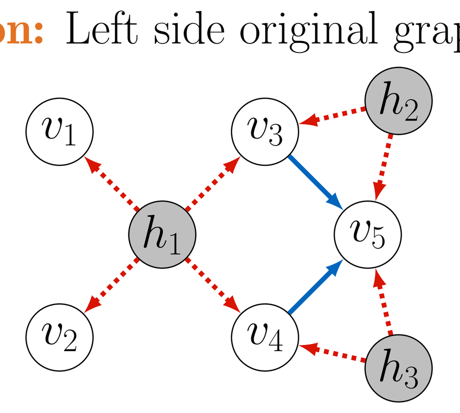


rationally identifiable

Latent projection: Left side original graph, right side its latent projection.



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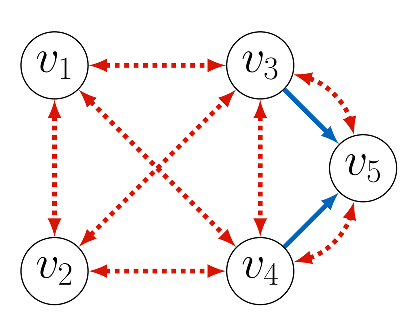

rationally identifiable

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